The Rev. Professor Graves, D.D., read the second part of his Paper on the solution of the equation of Laplace's functions.

"In the former part of this Paper I showed that the symbol

$$e^{x(jD_2+kD_3)},$$

or π, as we have denoted it for the sake of brevity, when operating upon $y^m z^n$, has the effect of changing it, if m and n be positive integers, into the

$$\left\{\frac{(m+n)\,!}{m\,!\;n\,!}\right\}^{th} \text{ part of the sum of all the } \frac{(m+n)\,!}{m\,!\;n\,!}$$

differently arranged products, of which each contains m factors equal to $y+jx$, and n equal to $z+kx$. But I reserved the consideration of the cases in which m and n were negative or fractional. In fact, I had ascertained by trial that the theorem just announced must undergo some modification in its statement before it could be extended to the case where m or n was negative; and I was at a loss to conceive what modification could render it applicable in the case where either of the exponents was fractional: the rule given for the formation of a *mean product* seeming of necessity to presume that the exponents were at least integer, if not positive numbers. In the present communication I desire to lay before the Academy the discussion of the reserved cases. In dealing with them I have been led so to modify my definition of a *mean product* as to make it apply where m and n are negative or fractional; at the same time that it coincides with my previous definition in the case where m and n are positive integers: and this has been accomplished by the help of mean products of i s, js, and ks, the fundamental theorems respecting which were stated at the end of my former Paper, p. 170. Thus it will be found that we are in possession of a complete and perfectly simple solution of the equation of Laplace's functions:—complete, as involving two arbitrary functions; and simple, as it is

disincumbered from all signs of differentiation or integration to be effected upon them. In this latter respect my solution seems to possess an advantage over those which have been given by Drs. Hargreave and Boole.

"I will now proceed briefly to describe the manner in which I investigate the effect of π upon $y^m z^n$, when m and n are negative or fractional.

"As before, I commence with particular and simple cases, expecting that the results will guide us, by the observation of analogies, to a general conclusion.

"Let us first calculate $\pi\ y^{-1}z^{-1}$. Putting $\rhd$ in place of $jD_2 + kD_3$, we have

$$\begin{aligned}
&\rhd\ y^{-1}z^{-1} = -jy^{-2}z^{-1} - ky^{-1}z^{-2},\\
&\rhd^2\ y^{-1}z^{-1} = -2\,!\,y^{-3}z^{-1} - 2\,!\,y^{-1}z^{-3},\\
&\rhd^3\ y^{-1}z^{-1} = 3\,!\,jy^{-4}z^{-1} + 2\,!\,ky^{-3}z^{-2} + 2\,!\,jy^{-2}z^{-3} + 3\,!\,ky^{-1}z^{-4},\\
&\rhd^4\ y^{-1}z^{-1} = 4\,!\,y^{-5}z^{-1} + 2\,.\,2\,!\,2\,!\,y^{-3}z^{-3} + 4\,!\,y^{-1}z^{-5},\\
&\rhd^5\ y^{-1}z^{-1} = -5\,!\,jy^{-6}z^{-1} - 4\,!\,ky^{-5}z^{-2} - 2\,.\,3\,!\,2\,!\,jy^{-4}z^{3}\\
&\qquad\qquad - 2\,.\,2\,!\,3\,!\,ky^{-3}z^{-4} - 4\,!\,jy^{-2}z^{-5} - 5\,!\,ky^{-1}z^{-6},\\
&\rhd^6\ y^{-1}z^{-1} = -6\,!\,y^{-7}z^{-1} - 3\,.\,4\,!\,2\,!\,y^{-5}z^{-3} - 3\,.\,2\,!\,4\,!\,y^{-3}z^{-5} - 6\,!\,y^{-1}z^{-7},\\
&\ldots\ldots = \ldots\ldots\ldots\ldots
\end{aligned}$$

Hence,

$$\left.\begin{aligned}
\pi\ y^{-1}z^{-1} = y^{-1}z^{-1} &- x\,(jy^{-2}z^{-1} + ky^{-1}z^{-2}) - \frac{x^2}{2\,!}(2\,!\,y^{-3}z^{-1} + 2\,!\,y^{-1}z^{-3})\\
&+ \frac{x^3}{3\,!}(3\,!\,jy^{-4}z^{-1} + 2\,!\,ky^{-3}z^{-2} + 2\,!\,jy^{-2}z^{-3} + 3\,!\,ky^{-1}z^{-4})\\
&+ \frac{x^4}{4\,!}(4\,!\,y^{-5}z^{-1} + 2\,.\,2\,!\,2\,!\,y^{-3}z^{-3} + 4\,!\,y^{-1}z^{-5})\\
&- \frac{x^5}{5\,!}(5\,!\,jy^{-6}z^{-1} + 4\,!\,ky^{-5}z^{-2} + 2\,.\,3\,!\,2\,!\,jy^{-4}z^{-3}\\
&\qquad + 2\,.\,2\,!\,3\,!\,ky^{-3}z^{-4} + 4\,!\,jy^{-2}z^{-5} + 5\,!\,ky^{-1}z^{-6})\\
&- \frac{x^6}{6\,!}(6\,!\,y^{-7}z^{-1} + 3\,.\,4\,!\,2\,!\,y^{-5}z^{-3} + 3\,.\,2\,!\,4\,!\,y^{-3}z^{-5} + 6\,!\,y^{-1}z^{-7})\\
&+ \&c. \ldots\ldots
\end{aligned}\right\}\text{(A)}$$

" Now let us compare this with

$$\tfrac{1}{2}\{(y+jx)^{-1}(z+kx)^{-1}+(z+kx)^{-1}(y+jx)^{-1}\},$$

to which the analogy of example (2) in my former Paper, p. 163, might lead us to expect to find it equal.

" Developing by the binomial theorem, we have

$$\left.\begin{aligned}(y+jx)^{-1}(z+kx)^{-1}=y^{-1}z^{-1}&-x\,(jy^{-2}z^{-1}+ky^{-1}z^{-2})\\&-x^2\,(y^{-3}z^{-1}-iy^{-2}z^{-2}+y^{-1}z^{-3})\\&+x^3\,(jy^{-4}z^{-1}+ky^{-3}z^{-2}+jy^{-2}z^{-3}+ky^{-1}z^{-4})\\&+x^4\,(y^{-5}z^{-1}-iy^{-4}z^{-2}+y^{-3}z^{-3}-iy^{-2}z^{-4}+y^{-1}z^{-5})\\&-x^5\,(jy^{-6}z^{-1}+ky^{-5}z^{-2}+jy^{-4}z^{-3}+ky^{-3}z^{-4}+jy^{-2}z^{-5}\\&\qquad+ky^{-1}z^{-6})\\&-x^6\,(y^{-7}z^{-1}-iy^{-6}z^{-2}+y^{-5}z^{-3}-iy^{-4}z^{-4}+y^{-3}z^{-5}\\&\qquad-iy^{-2}z^{-6}+y^{-1}z^{-7})\\&+\&c.\;\ldots\ldots\end{aligned}\right\}\text{(B)}$$

And $(z+kx)^{-1}(y+jx)^{-1}$ differs from this only in the signs of the terms containing i. Consequently, the development of

$$\tfrac{1}{2}\{y+jx)^{-1}(z+kx)^{-1}+(z+kx)^{-1}(y+jx)^{-1}\}$$

differs from the series just given only by the omission of these terms. But this omission will not make it agree with the expression already found for $\pi\ y^{-1}z^{-1}$.

" The discrepancy first shows itself in the numerical coefficients of the terms

$$x^3y^{-3}z^{-2},\ x^3y^{-2}z^{-3},\ \text{and}\ x^4y^{-3}z^{-3}.$$

In the former development (A) these coefficients are all $=\frac{1}{3}$. In the latter (B) to unity.

" Again, the coefficients of $x^5y^{-4}z^{-3}$, $x^5y^{-3}z^{-4}$, $x^5y^{-2}z^{-5}$, $x^5y^{-5}z^{-2}$, $x^6y^{-5}z^{-3}$, and $x^6y^{-3}z^{-5}$ are all equal in (A) to $\frac{1}{3}$, in (B) to unity. It is needless to proceed further in the comparison of the two developments.

" As regards the first instance of disagreement, viz. that between the coefficients in the two series of the terms $x^3y^{-3}z^{-2}$ and $x^3y^{-2}z^{-3}$; it must be observed that in (B) these terms have

respectively the imaginary coefficients k and j; or, more exactly, after the restoration of the powers of j and k suppressed in virtue of the equations $j^2 = k^2 = -1$; the imaginary coefficients $-j^2k$ and $-k^2j$. Now, by the theorems in p. 170, the mean value of the product of two j s and one k is,

$$\frac{2!\,1}{3!}\,\Sigma(2, 1) = \frac{2!\,1}{3!}(-1)\,k = -\frac{1}{3}\,k\,;$$

and the mean value of the product of two k s and one j is $-\frac{1}{3}j$. So that, so far as concerns the terms $x^3y^{-3}z^{-2}$ and $x^3y^{-2}z^{-3}$, the difference between the two developments consists in this: that in (B) these terms are multiplied by ordinary products, but in (A) by mean products of j s and k s.

"The next discrepancy noticeable is in the coefficient of $x^4y^{-3}z^{-3}$. In (B) this is j^2k^2, if the suppressed powers of j and k be restored. Now the mean value of the product of two j s and two k s, by the formulæ of p. 170, is

$$\frac{2!\,2!}{4!}\cdot\frac{2!}{1\,.\,1} = \frac{1}{3}.$$

Here again we find a mean product of j s and k s in (A), corresponding to an ordinary product in (B).

"The next discrepancy occurs in the case of the coefficients of $x^5y^{-4}z^{-3}$ and $x^5y^{-3}z^{-4}$. In (B) these are $-j^3k^2$, and $-j^2k^3$, if we restore the suppressed powers of j and k. Now the mean value of the product of three j s and two k s is,

$$\frac{3!\,2!}{5!}\cdot\frac{2!}{1\,.\,1}\,j, \text{ or } \frac{1}{5}\,j\,;$$

and the mean value of the product of two j s and three k s is $\frac{1}{5}\,k$. Here again, therefore, we find mean values of products of j s and k s in (A), corresponding to ordinary products in (B).

"Let us next consider the coefficients of $x^5y^{-5}z^{-2}$ and $x^5y^{-2}z^{-5}$. In (B) they are $-j^4k$ and $-jk^4$: but the mean values of pro-

ducts of one j and four k s, or four j s and one k, are respectively

$$\frac{4!\,1}{5!}j \text{ and } \frac{4!\,1}{5!}k, \text{ or } \frac{1}{5}j \text{ and } \frac{1}{5}k,$$

so that here likewise we find mean products in (A) standing in place of ordinary products in (B).

"Lastly, the coefficients of $x^6y^{-5}z^{-3}$, and $x^6y^{-3}z^{-5}$ in (B) are j^4k^2 and j^2k^4. Now the mean value of the product of four j s and two k s, or of two j s and four k s, is $= -\frac{1}{5}$, which is the coefficient belonging to $x^6y^{-5}z^{-3}$ and $x^6y^{-3}z^{-5}$ in (A).

"It is, moreover, to be observed that all the terms which disappear out of (B) have coefficients like jk^3, the exponents of both j and k being odd numbers. Now the mean value of a product containing odd numbers both of j s and k s has been proved equal to 0.

"It is also deserving of remark, that where the developments coincide, the mean values and the ordinary products are equal. In fact, these coincidences occur in the case of the first and last terms in each group of terms multiplied by the same power of x; and in their coefficients j s and k s are not combined.

"So far, then, as our examination has extended, the discrepancy between the developments (A) and (B) consists in this, that mean values of products of j s and k s stand in the former in place of ordinary products occurring in the latter.

"The careful examination of this one example led me to suspect, that when m and n are integers, the difference between the expression $\pi\ y^{-m}z^{-n}$ and the ordinary algebraic development of $(y+jx)^{-m}\,(z+kx)^{-n}$, effected without any regard to the properties of j and k, consists merely in this, that mean products of j s and k s take the place in the former of ordinary products occurring in the latter. To test this hypothesis let us try another simple example, in which y and z are not symmetrically involved. Let us calculate $\pi\ yz^{-1}$. We shall have then

$$\triangleright\ yz^{-1} = jz^{-1} - kyz^{-2},$$
$$\triangleright^2\ yz^{-1} = -2\,!\,yz^{-3},$$
$$\triangleright^3\ yz^{-1} = -2\,!\,jz^{-3} + 3\,!\,kyz^{-4},$$
$$\triangleright^4\ yz^{-1} = 4\,!\,yz^{-5},$$
$$\triangleright^5\ yz^{-1} = 4\,!\,jz^{-5} - 5\,!\,kyz^{-6},$$
$$\triangleright^6\ yz^{-1} = -6\,!\,yz^{-7},$$
$$\ldots\ldots = \ldots\ldots$$

Hence, we find

$$\begin{aligned}\pi\ yz^{-1} = yz^{-1} + x\,(jz^{-1} - kyz^{-2}) - x^2yz^{-3} - \frac{x^3}{3\,!}\,(2\,!\,jz^{-3} - 3\,!\,kyz^{-4}) \\ + x^4yz^{-5} + \frac{x^5}{5\,!}\,(4\,!\,jz^{-5} - 5\,!\,kyz^{-6}) - x^6yz^{-7} - \&c.\end{aligned} \tag{A}$$

Now let us compare this with the development of

$$(y + jx)\ (z + kx)^{-1}.$$

Expanding by the binomial theorem, and preserving the powers of j and k, when both appear in the same coefficient, we have

$$\begin{aligned}(y + jx)\ (z + kx)^{-1} = yz^{-1} + x\,(jz^{-1} - kyz^{-2}) - x^2\,(jkz^{-2} + yz^{-3}) \\ + x^3\,(jk^2z^{-3} + kyz^{-4}) - x^4\,(jk^3z^{-4} - yz^{-5}) \\ + x^5\,(jk^4z^{-5} - kyz^{-6}) - x^6\,(jk^5z^{-6} + yz^{-7})\ \ldots \\ + \&c.\ \ldots\ldots\end{aligned} \tag{B}$$

The discrepancies between the developments (A) and (B) are numerous, but all of them are of the same kind. In the first place the terms x^2z^{-2}, x^4z^{-4}, x^6z^{-6}, &c., do not appear in (A). In (B) they have the coefficients jk, jk^3, jk^5, &c. But the *mean* values of such products of j and k are equal to zero.

"Again, the mean value of the product of one j and 2ν k s is, $\dfrac{1}{2\nu + 1}\,(-1)^\nu$. Hence the coefficients in the two developments of x^3z^{-3}, x^5z^{-5}, &c., differ just in this: that in (A) they are mean products, in (B) ordinary products of j s and k s. Thus it appears, as we anticipated, that if we substitute mean products of j s and k s for ordinary products throughout the

entire development of $(y+jx)\ (z+kx)^{-1}$, we shall produce the development of $\pi\ yz^{-1}$.

"Without stopping to consider the case where m or n is fractional, we may now proceed to establish the mode of interpreting $\pi\ y^m z^n$, whatever be the nature of m and n.

"The coefficient of $x^{\mu+\nu}\, y^{m-\mu}\, z^{n-\nu}$ in the development of $\pi\ y^m z^n$, is equal to the coefficient of $x_{\mu}{}^{+}{}_{\nu} D_2^{\mu} D_3^{\nu}$ in the development of $e^{x(jD_2+kD_3)}$, multiplied by

$$m\,(m-1)\ldots\ldots(m-\mu+1)\,n\,(n-1)\ldots\ldots(n-\nu+1).$$

But, in the development of the exponential, $D_2^{\mu} D_3^{\nu}$ occurs only in the term

$$\frac{x^{\mu+\nu}(jD_2+kD_3)^{\mu+\nu}}{(\mu+\nu)\,!},$$

and there has for its coefficient

$$\frac{\Sigma\,(\mu,\ \nu)}{(\mu+\nu)\,!}.$$

Consequently, the coefficient sought is

$$\frac{m\,(m-1)\ldots\ldots(m-\mu+1)\,n\,(n-1)\ldots\ldots(n-\nu+1)}{(\mu+\nu)\,!}\,\Sigma\,(\mu,\ \nu).$$

"But again, if we develop $(y+jx)^m\ (z+kx)^n$ in the manner already mentioned, that is to say, preserving all the powers of j and k, and afterwards substituting mean products for ordinary products of these imaginaries; the coefficient of $x^{\mu+\nu}y^{m-\mu}\, z^{n-\nu}$ is plainly

$$\frac{m\,(m-1)\ldots\ldots(m-\mu+1)\,n\,(n-1)\ldots\ldots(n-\nu+1)}{\mu\,!\ \nu\,!}\,M(\overset{\mu}{j},\ \overset{\nu}{k}),$$

or, since

$$M(\overset{\mu}{j},\ \overset{\nu}{k})=\frac{\mu\,!\,\nu\,!}{(\mu+\nu)\,!}\,\Sigma\,(\mu,\ \nu),$$

to

$$\frac{m\,(m-1)\ldots\ldots(m-\mu+1)\,n\,(n-1)\ldots\ldots(n-\nu+1)}{(\mu+\nu)\,!}\,\Sigma\,(\mu,\ \nu).$$

Thus, we have demonstrated generally that the expression

$$e^{x(jD_2+kD_3)} f(y, z),$$

is to be interpreted as follows:—

"Substitute $y+jx$ for y, and $z+kx$ for z in $f(y, z)$; taking care to leave all powers of j and k in evidence, and then replace all the products of j s and k s, obtained in this way by *mean products* of those imaginaries.

"Reasoning and processes in all respects similar lead to the conclusion that the effect of the symbol

$$e^{w(iD_1+jD_2+kD_3)}$$

upon any function whatsoever of x, y, and z will be to change it into the same function, *in its mean state*, of $x+iw$, $y+jw$, and $z+kw$. By this it is to be understood, that after this change of the variables has been made, and the development effected as if i, j, and k were ordinary algebraic quantities, *mean values* of products of the imaginaries are to be substituted for ordinary ones.

"Reverting now to the solutions of the differential equations noticed in the first part of this Paper, p. 168, we see that they hold good, without any limitation of the nature of the arbitrary functions, provided we modify, or rather perfect, our conception of the *mean state* of a function in the manner just described.

"Our new definition of a mean product, or of a mean function, coincides with that given at p. 166, in the case where m and n are positive integers; and it includes the cases where m and n are negative or fractional, to which the original definition of a mean product is inapplicable.

"If it should prove that the solution of Laplace's equation now attained to, viz.:

$$V = Mf_1(y+jx, z+kx) + Mf_2(y-jx, z-kx),$$

is something more than a mathematical curiosity, and answers the demands of physical inquiry, we shall have reason to rejoice not only in the fruits of that particular discovery, but

also in the anticipation that other important steps in mathematics may be made by the help of Sir William Hamilton's imaginaries. I hope before long to be able to furnish the Academy with some reply to the questions here suggested.

Rev. Samuel Haughton made some observations on the Rev. Dr. Graves' paper.

Rev. Humphrey Lloyd, D.D., read a further communication "on the magnetic influence of the Moon."

The President and Rev. Samuel Haughton made some remarks, eliciting explanations from Dr. Lloyd as to the analogy of the magnetic phenomena described by him, to corresponding phenomena connected with the tides.

In the absence of Edward J. Cooper, Esq., his Paper on "Ecliptic Catalogues" was read by the Secretary:—

"Having completed the catalogues of ecliptic stars observed here during six years, it occurred to me to employ a few holidays, which I gave myself after the publication of our third volume, in examining some of the general results deducible from them, and comparing these results with concurrent meteorological phenomena. My object was to ascertain the soundness of a preconceived opinion, that the records of the state of the weather are useless as a guide in estimating the most favourable periods of the year for astronomical observations. To the investigation I added a search for any striking facts that might appear during the course of the work, in which Mr. Graham has been the principal performer, in the capacity of an indefatigable observer.

"The mode of proceeding which we originally adopted was, as is stated in the Introduction to the first volume of 'Ecliptic Stars,' one which we considered the most likely to